公式表

Laplace算符

$$\nabla^2 = \begin{cases} \dfrac{1}{r^2}\dfrac{\partial}{\partial r}\left(r^2\dfrac{\partial}{\partial r}\right) + \dfrac{1}{r^2\sin\theta}\dfrac{\partial}{\partial\theta}\left(\sin\theta\dfrac{\partial}{\partial\theta}\right) + \dfrac{1}{r^2\sin^2\theta}\dfrac{\partial^2}{\partial\phi^2} & \text{球坐标系} \\ \dfrac{1}{r}\dfrac{\partial}{\partial r}\left(r\dfrac{\partial}{\partial r}\right) + \dfrac{1}{r^2}\dfrac{\partial^2}{\partial\theta^2} + \dfrac{\partial^2}{\partial z^2} & \text{柱坐标系} \end{cases}$$

Legendre多项式

$$P_l(x) = \sum_{r=0}^{[l/2]} \frac{(-1)^r}{2^l r!} \frac{(2l-2r)!}{(l-r)!(l-2r)!} x^{l-2r}$$

微分表示

$$P_l(x) = \frac{1}{2^l l!}\frac{\mathrm{d}^l}{\mathrm{d}x^l}(x^2-1)^l$$

生成函数

$$\frac{1}{\sqrt{1-2xt+t^2}} = \sum_{l=0}^{\infty} P_l(x)t^l$$

递推关系

$$(2l+1)xP_l(x) = (l+1)P_{l+1}(x) + lP_{l-1}(x)$$

正交关系

$$\int_{-1}^{1} P_l(x)P_k(x)\mathrm{d}x = \frac{2}{2l+1}\delta_{lk}$$

连带Legendre函数

$$P_l^m(x) = (-1)^m(1-x^2)^{m/2}\frac{\mathrm{d}^m}{\mathrm{d}x^m}P_l(x)$$

正交关系

$$\int_{-1}^{1} P_l^m(x)P_k^m(x)\mathrm{d}x = \frac{(l+m)!}{(l-m)!}\frac{2}{2l+1}\delta_{lk}$$

归一化的球面调和函数

$$Y_l^m(\theta,\phi) = \sqrt{\frac{(l-|m|)!}{(l+|m|)!}\frac{2l+1}{4\pi}}P_l^{|m|}(\cos\theta)\mathrm{e}^{\mathrm{i}m\phi}$$

Bessel函数

$$J_\nu(x) = \sum_{k=0}^{\infty}\frac{(-1)^k}{k!\Gamma(k+\nu+1)}\left(\frac{x}{2}\right)^{2k+\nu}$$

积分公式

$$\int J_\nu^2(x)x\mathrm{d}x = \frac{1}{2}x^2J_\nu'^2(x) + \frac{x^2-\nu^2}{2}J_\nu^2(x) + C$$

Neumann函数

$$N_\nu(x) = \frac{\cos\nu\pi J_\nu(x) - J_{-\nu}(x)}{\sin\nu\pi}$$

递推关系

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}[x^{\nu}J_{\nu}(x)] &= x^{\nu}J_{\nu-1}(x) \\
\frac{\mathrm{d}}{\mathrm{d}x}[x^{-\nu}J_{\nu}(x)] &= -x^{-\nu}J_{\nu+1}(x) \\
J_{\nu-1}(x) - J_{\nu+1}(x) &= 2J'_{\nu}(x) \\
J_{\nu-1}(x) + J_{\nu+1}(x) &= \frac{2\nu}{x}J_{\nu}(x)
\end{aligned}$$

$x \to 0$

$$J_n(x) \sim \frac{1}{n!}\left(\frac{x}{2}\right)^n$$

$$N_0(x) \sim \frac{2}{\pi}\ln\frac{x}{2}, \quad N_n(x) \sim -\frac{(n-1)!}{\pi}\left(\frac{x}{2}\right)^{-n}$$

$x \to \infty$

$$J_{\nu}(x) \sim \sqrt{\frac{2}{\pi x}}\cos\left(x - \frac{\nu\pi}{2} - \frac{\pi}{4}\right)$$

$$N_{\nu}(x) \sim \sqrt{\frac{2}{\pi x}}\sin\left(x - \frac{\nu\pi}{2} - \frac{\pi}{4}\right)$$

生成函数

$$\exp\left\{\frac{x}{2}(t - t^{-1})\right\} = \sum_{n=-\infty}^{\infty} J_n(x)t^n$$

球Bessel函数

$$j_l(x) = \sqrt{\frac{\pi}{2x}}J_{l+1/2}(x) = x^l\left(-\frac{\mathrm{d}}{x\mathrm{d}x}\right)^l\left\{\frac{\sin x}{x}\right\}$$

$$n_l(x) = \sqrt{\frac{\pi}{2x}}N_{l+1/2}(x) = x^l\left(-\frac{\mathrm{d}}{x\mathrm{d}x}\right)^l\left\{-\frac{\cos x}{x}\right\}$$

递推关系

$$\begin{aligned}
\frac{\mathrm{d}}{\mathrm{d}x}[x^{l+1}j_l(x)] &= x^{l+1}j_{l-1}(x) \\
\frac{\mathrm{d}}{\mathrm{d}x}[x^{-l}j_l(x)] &= -x^{-l}j_{l+1}(x)
\end{aligned}$$